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Long-time behaviour of nonautonomous SPDE's

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Abstract

It is proved that under suitable conditions the probability laws of two arbitrary solutions of the infinite dimensional stochastic equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + f(t,X_t)\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t$$

converge to each other, as time goes to infinity, in the strong (variational) topology. To this end, some lower estimates on the transition density of the solution, with respect to a certain Gaussian measure, are obtained. In addition, an explicit formula for the density is given, in the case where $Q^{-1/2}f$ is bounded. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In this paper, dynamics of probability laws of solutions to the infinite-dimensional stochastic equation of the form

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + f(t,X_t)\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t$$

are studied in the space of probability measures endowed with the metric of total variation of measures, where A is an unbounded linear operator on a separable real Hilbert space H, the nonlinear term $f:\mathbb{R}_+\times H\to H$ is measurable, $Q^{1/2}$ is a bounded operator on H and W_t is a cylindrical Wiener process on H.

Under suitable conditions yielding a kind of ultimate mean-square boundedness of solutions and sufficient nondegeneracy of the noise it is proved that the probability laws of two arbitrary solutions converge to each other as time tends to infinity, in the strong (variational) topology. To achieve this goal some results on densities of the transition probabilities of the solution with respect to a certain Gaussian measure are

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also obtained; for example, a lower estimate on the density and (for bounded nonlinear terms) an explicit formula for the density, which may be of independent interest.

Large-time behaviour of solutions has been broadly investigated for autonomous SPDE's that induce homogeneous Markov processes in Hilbert or Banach spaces, especially in connection with problems of existence and uniqueness of invariant measures and ergodic properties of the associated Markov transition semigroups. For stochastic reaction–diffusion equations perturbed by the space–time white noise, ergodic results based on the strong Feller property and topological irreducibility have been established in Maslowski (1989, 1993) and in Manthey and Maslowski (1992) by a method going back essentially to Khas'miskii (1960) (for nonlocally compact spaces cf. also Seidler, 1997). These results have been extended by Da Prato and Zabczyk (1992, 1996) (and references therein), Chojnowska-Michalik and Goldys (1995), Gątarek and Goldys (1997), Cerrai (1998) (see also the references therein), as a consequence of regularity properties of transition Markov semigroups. An alternative method based on the Elworthy formula has been used in Da Prato et al. (1993) and in Peszat and Zabczyk (1995). All the above papers concern abstract evolution equations and are applicable basically to stochastic reaction–diffusion equations. Analogous results have been obtained for the Burgers equation in Da Prato and Gątarek (1995), for the stochastic Cahn–Hilliard equation in Da Prato and Debussche (1996) and for the two-dimensional stochastic Navier–Stokes equation in Flandoli and Maslowski (1995) and in Ferrario (1997). Another method of the proof of the strong Feller property has been developed and applied to stochastic delay systems and to stochastic parabolic PDE's in Maslowski and Seidler (1999).

In Jacquot and Royer (1995a, b) a general theory of Markov operators was used to prove geometric ergodicity (a kind of strong exponential stability of the invariant measure) for a particular but important stochastic parabolic equation. In Mueller (1993) the coupling techniques were used to prove strong exponential stability of the invariant measure for a nonlinear heat equation defined on a circle. Recently, in Shardlow (1999) geometric ergodicity for some SPDE's was proved.

Very little seems to be known in the case of nonautonomous SPDE's where the standard methods of ergodic theory are no longer available. The lower bound measure method which is used in the present paper has already been applied in a previous paper by the authors (Maslowski and Simão, 1997). The methodology has been basically inherited from the theory of deterministic discrete-time dynamical systems (see, e.g., Lasota and Mackey, 1994) and Markov chains.

The present paper extends the result of Maslowski and Simão (1997) in a substantial way. In Maslowski and Simão (1997), it was assumed that $Q = I$ and A is self-adjoint; this was important for the proofs based on finite-dimensional approximations. The nonlinear term f was assumed to be Lipschitz continuous with the range contained in some subspace of H which did not allow applications to semilinear equations of second order in space. These restrictions are removed in the present paper by means of some new infinite-dimensional techniques for the pinned (conditioned) Ornstein–Uhlenbeck process.

The method used in the paper could be basically extended to the case when the generator $A = A(t)$ is time-dependent and generates a two-parameter evolution operator.

However, some more technical hypotheses would make the presentation of the results much more complicated (for example, we would have to use the concept of a system of lower measures rather than one lower measure). Other generalizations of the equation (random coefficients, multiplicative noise term, etc.) probably cannot be considered without a deep modification of this technique. Let us note that limit behaviour of such general system has been intensively studied from other points of view (stability, global attractors, see Chueshov and Vuillermot, 1998, 2000; Leon and Nualart, 1998; Nualart and Viens, 2000).

The paper is divided into two sections. At the beginning of Section 1 assumptions on the particular terms of the equation are formulated. Then the results of the paper are stated and some examples are given. In Example 1.1 the assumptions are verified in the so-called commutative case. In Remark 1.1 and Example 1.2 the general results are applied to the case of a one-dimensional stochastic parabolic equation. Proposition 1.1 summarizes the results on the pinned (conditioned) process, that is, an Ornstein–Uhlenbeck process that is conditioned to go to a given point at terminal time. A linear nonautonomous equation for the process is derived and some regularity properties are established. In Propositions 1.2 and 1.3 some lower estimates on the density of the Markov transition probability with respect to certain Gaussian measures are given. In some cases a formula for the density is obtained (Proposition 1.2), which can be of independent interest (cf. Fuhrman, 1996).

The main result of the paper is Theorem 1.4 where the strong asymptotic stability of the adjoint Markov evolution operator is stated. In the homogeneous case (i.e., when f does not depend on t) our result automatically implies the existence of an invariant measure which therefore must be unique and globally asymptotically stable. An application to a system of non-autonomous stochastic semilinear parabolic equations is given in Example 1.2.

Section 2 contains the proofs of the above results.

Given Hilbert spaces Y and Z, we denote by $\mathscr{L}(Y,Z)$, $\mathscr{L}_1(Y,Z)$ and $\mathscr{L}_2(Y,Z)$, the spaces of bounded, trace class, and Hilbert–Schmidt linear operators $Y \to Z$, respectively, and we set $\mathscr{L}(Y)=\mathscr{L}(Y,Y)$, $\mathscr{L}_1(Y)=\mathscr{L}_1(Y,Y)$ and $\mathscr{L}_2(Y)=\mathscr{L}_2(Y,Y)$. Furthermore, if a densely defined linear operator is extendable to a bounded operator defined on the whole space, the extension is denoted by the same symbol if there is no danger of confusion.

1. Main results

Consider the equation

$$\begin{aligned} &\mathrm{d}X_t = (AX_t + f(t,X_t))\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t, \\ &X_0 = x, \end{aligned} \tag{1.1}$$

on a separable space $H=(H,\langle\cdot,\cdot\rangle,|\cdot|)$ where $x\in H$, W_t is a cylindrical Wiener process on H defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, Q is a positive bounded self-adjoint operator on H, and A generates an analytic semigroup S_t on H. Assume that $\langle Ax,x\rangle \leqslant -\omega|x|^2$, $x\in \mathrm{Dom}(A)$ for some $\omega>0$ and denote by $H_\delta=(H_\delta,|\,.\,|_\delta)$ the space

$\mathrm{Dom}((-A)^\delta)$ equipped with the graph norm $|\cdot|_\delta = |(-A)^\delta \cdot|$. Furthermore, $f:\mathbb{R}_+ \times H \to H$ is a measurable function satisfying $\mathrm{Im}\, f \subset \mathrm{Range}(Q^{1/2})$ and

(A.1) $\quad |Q^{-1/2}f(t,x)| \leqslant K(1+|x|_\delta), \quad x\in H_\delta,\ t\in\mathbb{R}_+,$

$$|f(t,x)| \leqslant K(1+|x|), \quad x\in H,\ t\in\mathbb{R}_+ \tag{1.3}$$

for some $K<\infty$ and $0<\delta<\frac{1}{2}$. We also consider the linear equation

$$\mathrm{d}Z_t = AZ_t\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t, \qquad Z_s = x,\ t\geqslant s,\ s\in\mathbb{R}_+ \tag{1.4}$$

and assume that there exists a $T>0$ such that

(A.2) $\quad \displaystyle\int_0^T t^{-\alpha}|S_tQ^{1/2}|^2_{\mathscr{L}_2(H)}\,\mathrm{d}t < \infty$

for some $\alpha > 2\delta$ where $\mathscr{L}_2$ denotes the space of Hilbert–Schmidt operators and δ is the same as in (A.1). By analyticity of the semigroup S_t and (A.2) we have

$$\int_0^T |S_tQ^{1/2}|^2_{\mathscr{L}_2(H,H_\delta)}\,\mathrm{d}t < \infty. \tag{1.5}$$

By (A.2) there exists a unique mild solution $Z_t = Z_t^{s,x}$ to (1.4) on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ whose law is concentrated on H_δ for each t and whose paths are elements of $C(\mathbb{R}_+;H_\delta)$ $\mathbb{P}$-a.s. Moreover, we have

$$\mathbb{E}|Z_t^{s,x}|^2_\delta \leqslant k_1(1+|x|^2_\delta), \quad x\in H_\delta,\ s,t\in\mathbb{R}_+ \tag{1.6}$$

for some $k_1<\infty$. It is also standard to see that if (A.1) and (A.2) hold true Eq. (1.1) has a (weakly unique) solution that can be obtained from the process $Z_t^{0,x}$ under the absolutely continuous change of probability

$$\frac{\mathrm{d}\mathbb{P}_x}{\mathrm{d}\mathbb{P}}(\omega) = \exp\left\{\int_0^T \langle Q^{-1/2}f(s,Z_s^{0,x}),\mathrm{d}W_s\rangle - \frac{1}{2}\int_0^T |Q^{-1/2}f(s,Z_s^{0,x})|^2\,\mathrm{d}s\right\}, \quad T\geqslant 0. \tag{1.7}$$

Both Eqs. (1.1) and (1.4) introduce Markov processes on H whose transition kernels we denote by $P(s,t,x,\cdot)$, and $Q(t,x,\cdot)$, respectively, that is

$$Q(t,x,\Gamma) = \mathbb{E}1_\Gamma(Z_t^{0,x}), \qquad P(s,t,x,\Gamma) = \mathbb{E}_{s,x}1_\Gamma(X_t),$$
$$x\in H,\ t\geqslant s\geqslant 0,\ \Gamma\in B(H), \tag{1.8}$$

where $\mathbb{E}_{s,x}$ is the expectation under the probability $\mathbb{P}_{s,x}$ corresponding to the initial condition $X_s = x$. Moreover, we have

$$Q(t,x,\cdot) = N(S_tx,Q_t) \quad \text{where } Q_t = \int_0^t S_rQS_r^*\,\mathrm{d}r.$$

If the process $Z^x = Z^{0,x}$ is strongly Feller, then the measure $P(s,t,x,\cdot)$, $t>s\geqslant 0$, $x\in H$, is equivalent to the measure $\mu := N(0,Q_\infty)$, which is the (unique) invariant measure for the process defined by (1.4). Our first aim is to find a representation and to prove a suitable lower estimate on the density

$$h_t(x,y) = \frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}\mu}(y). \tag{1.9}$$

In the sequel, we selectively use the assumptions below.

(A.3) Range $(S_t) \subset$ Range (Q_∞), $\quad t > 0$.

Assumption (A.3) is a stronger version of the usual hypoellipticity condition yielding the strong Feller property. While the strong Feller property is equivalent to the absolute continuity of $Q(t,x,\cdot)$, (A.3) is needed for sufficient regularity of the density. Obviously, (A.3) implies that the operator $S_t^* Q_t^{-1}$ is well defined and extendable so that $S_t^* Q_t^{-1} \in \mathscr{L}(H)$ for $t > 0$. We shall assume

(A.4) $|Q^{1/2} S_t^* Q_t^{-1}|_{\mathscr{L}(H)} \leqslant ct^{-1}, \quad t \in (0,1]$

for a constant $c > 0$.

Our last assumption concerning the linear part of Eq. (1.1) can be formulated in terms of the semigroup

$$\tilde{S}_t := Q_1 S_t^* Q_1^{-1}, \quad t \geqslant 0.$$

By (A.3) it is obvious that $\tilde{S}_t$ is a well defined semigroup of bounded operators on H. We shall assume

(A.5) The semigroup $\tilde{S}_t$ is analytic on H and the domain Dom$(\tilde{A})$ of its infinitesimal generator $\tilde{A}$ is isomorphic to Dom(A).

Conditions (A.2)–(A.5) are easy to verify in the commutative case, that is if we have the following diagonality property:

(D) $\exists\{e_i\}$ ONB in H, $\exists\ 0 < \alpha_i \to \infty,\ 0 < \lambda_i \leqslant \lambda_\infty$;

$$Ae_i = -\alpha_i e_i, \quad Qe_i = \lambda_i e_i, \quad i \in \mathbb{N}.$$

If the diagonality condition (D) holds true then the above assumptions (A.2)–(A.4) can be expressed in terms of $(\alpha_i), (\lambda_i)$: assumption (A.2) can be replaced by (1.5) because the $\mathbb{P}$-a.s. continuity of paths of $Z_t^{s,x}$ in $H_{\delta'}$ holds true for each $\delta' < \delta$ if conditions (1.5) and (D) are satisfied (cf. Iscoe et al., 1990). Assumption (A.5) is always satisfied.

Condition (1.5) is equivalent to

$$\sum_i \frac{\lambda_i}{\alpha_i^{1-2\delta}} < \infty, \tag{1.11}$$

assumption (A.3) reads

$$\exists C_t < \infty \quad \text{such that } \frac{\alpha_i^2}{\lambda_i^2} \mathrm{e}^{-2\alpha_i t} \leqslant C_t \quad \text{for each } t > 0 \tag{1.12}$$

and assumption (A.4) holds if and only if

$$\lambda_i \geqq \lambda_0 > 0, \tag{1.13}$$

(see also Remark 1.1).

Note that (1.12) (and, thereby, (A.3)) is equivalent to the strong Feller property in the present case.

Remark 1.1. Note that assumption (A.4) can be further relaxed to

$$\text{(A.4)} \quad |QS_t^* Q_t^{-1}|_{\mathscr{L}(H)} \leqslant ct^{-1}, \quad t \in (0,1]$$

if (A.1) is modified as follows:

$$f \text{ maps } \mathbb{R}_+ \times H_\delta \text{ into Range}(Q) \quad \text{and} \quad |Q^{-1} f(t,x)| \leqslant k(1 + |x|_\delta),$$
$$x \in H_\delta, \ t \in \mathbb{R}_+. \tag{1.14}$$

It is obvious that in the commutative case when (D) is satisfied (A.4)′ always holds (there is no restriction on the sequence (λ_i)). Therefore, it may be of interest to verify (1.14) in some concrete examples.

Assume that $f : H_\delta \to H_\delta$,

$$|f(x)|_\delta \leqslant k(1 + |x|_\delta), \quad x \in H_\delta \tag{1.15}$$

for some $k < \infty$ and Range$(Q) = H_\Delta$, for some $\Delta \geqslant 0$ (for simplicity we supress the dependence of f on t in the notation). Then $Q^{-1}(-A)^{-\Delta} \in \mathscr{L}(H)$, hence

$$|Q^{-1} f(x)| \leqslant |Q^{-1}(-A)^{-\Delta}|_{\mathscr{L}(H)} |f(x)|_\Delta \leqslant k(1 + |x|_\delta), \tag{1.16}$$

provided

$$\Delta \leqslant \delta \tag{1.17}$$

holds.

For example, if $H = L^2(D)$, D denotes a bounded domain in $\mathbb{R}^d$, with a regular boundary, $A = \Delta$, Dom$(A) = H^2(D) \cap H_0^1(D)$, and $f : H \to H$, $f(x)(\xi) = F(x(\xi))$, $\xi \in D$, for some $F : \mathbb{R} \to \mathbb{R}$, then the norm H_δ is equivalent to the norm of Sobolev–Slobodetskii space $H^{2\delta}(D)$. Hence for $0 < \delta < \frac{1}{2}$ we have

$$\begin{aligned} |f(x)|_\delta^2 &\leqslant \text{const} \left(|F(x(\cdot))|_{L^2(D)}^2 + \int\!\!\int_{D \times D} \frac{|F(x(\xi)) - F(x(\eta))|^2}{|\xi - \eta|^{d+2\delta}} \, \mathrm{d}\xi \, \mathrm{d}\eta \right) \\ &\leqslant \text{const}\,(1 + |x|_\delta^2) \end{aligned} \tag{1.18}$$

for $x \in H_\delta$ if F is (globally) Lipschitz continuous.

Example 1.1. Let us further specify the assumptions in the diagonal case when (D) is assumed if

$$\text{Range}\,(Q) = H_\Delta \tag{1.19}$$

for some $\Delta \in (0, \delta)$. In this case we also have

$$\text{Range}\,(Q^{1/2}) = H_{\Delta/2}. \tag{1.20}$$

As noted above conditions (A.2), (A.4)′ and (A.5) are always satisfied; (A.3) is equivalent to (1.12). Thus, if f satisfies the stronger condition (1.14) it only remains to verify the regularity condition (1.5). By (1.20) it is obviously satisfied if and only if

$$\sum_i \frac{1}{\alpha_i^{1-2\delta+\Delta}} < \infty. \tag{1.21}$$

For instance, if $A=\partial/\partial\xi^2$ for $\xi\in(0,1)$, $\mathrm{Dom}(A)=H^2(0,1)\cap H_0^1(0,1)$, then $\alpha_i\sim -i^2$, so (1.21) is satisfied if

$$\Delta > 2\delta - \tfrac{1}{2}. \tag{1.22}$$

In other words, if $\mathrm{Range}(Q)=H_\Delta$, then our assumptions can be satisfied by a suitable choice of δ for every $\Delta\in[0,\frac{1}{2})$ (see also Example 1.2 below).

We shall introduce the concept of pinned (or conditioned) Ornstein–Uhlenbeck (O–U) process. The O–U process $Z^{s,x}$ defined by Eq. (1.4) has almost surely H-continuous paths. Let $g_\Phi : H\to\mathbb{R}$ be a regular version of the conditional expectation $\mathbb{E}[\Phi(Z^{s,x})\,|\,Z^{s,x}_{s+1}]$ for a $P_{Z^{s,x}}$-integrable functional $\Phi : C([s,s+1];H)\to\mathbb{R}$, where $P_{Z^{s,x}}$ is the law of $Z^{s,x}$ on $C([s,s+1],H)$. That is, we have

$$g_\Phi(Z^{s,x}_{s+1})=\mathbb{E}[\Phi(Z^{s,x})\,|\,Z^{s,x}_{s+1}],\quad \mathbb{P}\text{-a.s.} \tag{1.23}$$

or, using the usual notation,

$$g_\Phi(y)=\mathbb{E}[\Phi(Z^{s,x})\,|\,Z^{s,x}_{s+1}=y]\quad \text{for } Q(1,x,\cdot)\text{-almost all } y\in H. \tag{1.24}$$

Definition 1.1. Given $x\in H$ we say that $(\hat{Z}^{x,y}_{s,t})$, $t\in[s,s+1]$, $y\in H$, is a pinned O–U process (conditioned to go from x at time $t=s$ to y at time $t=s+1$) if for each $\Phi\in L^1(C([s,s+1],H);P_{Z^{s,x}})$ we have

$$g_\Phi(y)=\mathbb{E}\Phi(\hat{Z}^{x,y})\quad \text{for } Q(1,x,\cdot)\text{-almost all } y\in H. \tag{1.25}$$

Consider the linear nonautonomous equation

$$\begin{aligned} d\hat{Z}_t &= (A\hat{Z}_t - QS^*_{1-t+s}Q^{-1}_{1-t+s}S_{1-t+s}\hat{Z}_t + QS^*_{1-t+s}Q^{-1}_{1-t+s}y)\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t,\\ &\quad t\in(s,s+1), \qquad\qquad (1.26)\\ \hat{Z}_s &= x\in H, \end{aligned}$$

where $x,y\in H$ are fixed. We will summarize our results now.

Proposition 1.1. *Assume* (A.2)–(A.5) *and let* $x,y\in H_\delta$. *Then*

(i) *Eq.* (1.26) *has a unique mild solution on the interval* $[s,s+1)$, *moreover, the process* $\hat{Z}_t$ *belongs to* $C([s,s+1],H_\delta)$ $\mathbb{P}$-*a.s. and*

$$\mathbb{E}|\hat{Z}_t|^2_\delta \leqslant k(1+|x|^2_\delta+|y|^2_\delta),\quad t\in(s,s+1),\ s\in\mathbb{R}_+,\ x,y\in H_\delta, \tag{1.27}$$

where the constant k *does not depend on the choice of* x,y,s *and* t.

(ii) $(\hat{Z}_t)=(\hat{Z}^{x,y}_t)$ *is the pinned process corresponding to the linear equation* (1.4). *The process* $\hat{Z}_t$ *has a version with paths a.s. in* $C^\lambda([s,s+1],H)$ *for each* $\lambda<\alpha$ *and we have*

$$\mathbb{E}|y-\hat{Z}_r|^2 \leqslant C_\delta(1+s-r)^{2\delta}(1+|y|^2_\delta+|x|^2_\delta),\quad x,y\in H_\delta,\ r\in[s,s+1] \tag{1.28}$$

for a constant C_δ.

Note that it is sufficient to define the pinned O–U process for $y\in H_\delta$ as $Q(1,x,H_\delta)=1$ for $x\in H_\delta$ by (1.6). Now set

$$\Psi(s,t,x,y):=\exp\left\{\int_s^t \langle Q^{-1/2}f(r,\hat{Z}_r),\mathrm{d}W_r\rangle - \frac{1}{2}\int_s^t |Q^{-1/2}f(r,\hat{Z}_r)|^2\,\mathrm{d}r\right.$$

$$- \int_s^t \langle Q^{-1/2} f(r,\hat{Z}_r), Q^{1/2} S^*_{1+s-r} Q^{-1}_{1+s-r} S_{1+s-r} \hat{Z}_r - Q^{1/2} S^*_{1+s-r} Q^{-1}_{1+s-r} y \rangle \, \mathrm{d}r \tag{1.29}$$

for $t \in [s, s+1]$, $s \geqslant 0$, $x, y \in H_\delta$.

Proposition 1.2. *Assume* (A.1)–(A.5) *and let* $x \in H_\delta$. *Then*

$$\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y) \geqslant \mathbb{E}\Psi(t,t+1,x,y) \tag{1.30}$$

holds for $Q(1,x,\cdot)$*-almost all* $y \in H_\delta$. *If, moreover,* $Q^{-1/2} f$ *is bounded then there is equality in* (1.30).

Formula (1.30) allows us to show some estimates on the density h (recall the definition (1.9)).

Proposition 1.3. *Assume* (A.1)–(A.5). *Then there is a constant* $k > 0$ *such that*

$$h_t(x,y) = \frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}\mu}(y) \geqslant k \exp\{-k(|x|^2_\delta + |y|^2_\delta)\} \tag{1.31}$$

for each $x \in H_\delta$, $t \in \mathbb{R}_+$ *and* $Q(1,x,\cdot)$ *almost all* $y \in H_\delta$.

Let $(P^*_{s,t})$, $0 \leqslant s \leqslant t$, be the family of adjoint Markov operators corresponding to Eq. (1.1). More precisely, denote by $\mathscr{P}$ the set of probability measures on the Borel sets $\mathscr{B}$ of H and set

$$[P^*_{s,t}\nu](\Gamma) := \int P(s,t,x,\Gamma)\nu(\mathrm{d}x), \quad 0 \leqslant s \leqslant t, \ \nu \in \mathscr{P}, \ \Gamma \in \mathscr{B}. \tag{1.32}$$

Clearly, $P^*_{s,t}\nu$ is the probability law of the solution X_t of Eq. (1.1) starting at time s with initial law ν.

Now we shall formulate our last assumption that is needed to verify a kind of "ultimate boundedness" of solutions of Eq. (1.1). Recall that $\langle Ax, x\rangle \leqslant -\omega |x|^2$, $x \in \mathrm{Dom}(A)$ and assume

(A.6) $\langle f(t, x+y), x\rangle \leqslant k(t)|x|^2 + a(t, |y|), \quad t \in \mathbb{R}_+, \ x, y \in H,$

where $k : \mathbb{R}_+ \to \mathbb{R}$, $a : \mathbb{R}^2_+ \to \mathbb{R}_+$ are measurable, locally bounded functions, $a(t,\cdot)$ is increasing for each $t \geqslant 0$. Moreover, $\tilde{\omega}(r) := \omega - k(r) \geqslant 0$ and

$$\exp\left\{-\int_s^t \tilde{\omega}(r)\,\mathrm{d}r\right\} \to 0, \quad t \to \infty,$$

$$\sup_{t \geqslant s} \int_s^t \exp\left\{-\int_r^t \tilde{\omega}(\lambda)\,\mathrm{d}\lambda\right\} \mathbb{E}a(r, |\Phi(r)|)\,\mathrm{d}r \leqslant M(s),$$

for each $s \geqslant 0$ and a constant $M(s) < \infty$ where

$$\Phi(r) = \int_s^r S_{r-u} Q^{1/2}\,\mathrm{d}W_u, \quad t \geqslant s.$$

Now we shall state our main result.

Theorem 1.4. *Assume* (A.1)–(A.6), *then*

$$\|P^*_{s,t}\nu_1 - \mathbb{P}^*_{s,t}\nu_2\|_{\mathrm{var}} \to 0 \quad \text{as } t \to \infty \tag{1.33}$$

for each $s \in \mathbb{R}_+$, $\nu_1, \nu_2 \in \mathscr{P}$. *If, moreover,* $f(t,x) = f(x)$ *does not depend on* t *then the Markov process induced by Eq.* (1.1) *is homogeneous* (*so we can set* $P^*_{s,t} = P^*_{t-s}$, $t \geqslant s \geqslant 0$), *there exists an invariant measure* μ^* *for Eq.* (1.1), *and*

$$\|P^*_t \nu - \mu^*\|_{\mathrm{var}} \to 0, \quad t \to \infty \tag{1.34}$$

for each $\nu \in \mathscr{P}$, *where* $\|\cdot\|_{\mathrm{var}}$ *denotes the norm of total variation of a measure.*

Example 1.2. Consider the system of parabolic SPDE's

$$\begin{aligned}
&\frac{\partial y_i}{\partial t}(t,\xi) = \frac{\partial^2 y_i}{\partial \xi^2}(t,\xi) + F_i(t, y(t,\xi)) + \eta_i(t,\xi), \quad i = 1,\ldots,d, \ (t,\xi) \in \mathbb{R}_+ \times (0,1),\\
&y_i(0,\xi) = x_i(\xi), \quad \xi \in (0,1),\\
&y_i(t,0) = y_i(t,1) = 0, \quad i = 1,2,\ldots,d,
\end{aligned} \tag{1.35}$$

where η_i, $i = 1,2,\ldots,d$ are stochastically independent space–time white noises and $F_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. System (1.35) can be rewritten in the usual way in the form

$$\mathrm{d}X_t = (AX_t + f(t,X_t))\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t, \quad X_0 = x,$$

on the Hilbert space $H = (L^2(0,1))^d$ where $Ay = (\partial^2 y_i/\partial \xi^2)$, $y \in \mathrm{Dom}(A) = (H^1_0(0,1) \cap H^2(0,1))^d$, W_t is a cylindrical Wiener process on H with covariance $Q = I$, and $f = (f_i)_{i=1,2,\ldots,d}$,

$$f_i(t,x)(\xi) := F_i(t, x(\xi)), \quad t \in \mathbb{R}_+, \ x \in H, \ \xi \in (0,1).$$

The function F is measurable, $F(t,\cdot)$ grows at most linearly, and

$$(F(t,\xi+\lambda),\xi)_{\mathbb{R}^d} \leqslant k|\xi|^2 + a|\lambda|^2, \quad \xi,\lambda \in \mathbb{R}^d, \ t \in \mathbb{R}_+$$

for some real constants a, k, $k < \pi$. Then all assumptions of Theorem 1.4 are satisfied as follows from (A.6) and the discussion preceding Remark 1.1. Moreover, from Example 1.1 and Remark 1.1 it follows that the result holds true even for "slightly degenerate" covariance operators Q. More precisely, if Q is such that the diagonality condition (D) holds and Range $(Q) = H_\Delta$ with $0 \leqslant \Delta < \frac{1}{2}$ and $F(t,\cdot)$ is globally Lipschitz continuous (with the Lipschitz constant independent of $t \in \mathbb{R}_+$) then Theorem 1.4 can be applied.

2. Proofs

Proof of Proposition 1.1. *Step I*: For simplicity we take $s = 0$. Set

$$H(t,z) = -QS^*_{1-t}Q^{-1}_{1-t}S_{1-t}z + QS^*_{1-t}Q^{-1}_{1-t}y, \quad t \in [0,T], \tag{2.1}$$

where $T \in (0,1)$ is fixed. By (A.4) the function $H : [0,T] \times H_\delta \to H$ is measurable and

$$|H(r,z)| \leqslant c_0|y|_\delta + \sup_{r\in[0,1]} |S_r|_{\mathscr{L}(H)} c_0 |z|_\delta \leqslant c_1|y|_\delta + c_2|z|_\delta \tag{2.2}$$

holds for $r \in [0,T]$ (with some c_0, c_1, c_2 possibly depending on T). By (1.6) we have

$$\sup_{t\in[0,1]} \mathbb{E}|Z_t|_\delta^2 \leqslant C, \tag{2.3}$$

where Z_t is the O–U process solving (1.4) with $Z_0 = 0$. Setting $\mathscr{B} = C([0,T], L^2(\Omega; H_\delta))$ and $\Lambda : \mathscr{B} \to \mathscr{B}$,

$$\Lambda(Y)(t) = S_t x + \int_0^t S_{t-r} H(r, Y(r) - Z_r)\,\mathrm{d}r, \quad t \in [0,T], \tag{2.4}$$

we have that

$$\begin{aligned}
\sup_{t\in[0,\tau]} \mathbb{E}|\Lambda(Y)(t)|_\delta^2 &\leqslant c_3|x|_\delta^2 + \sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \frac{c_4}{(t-r)^\delta}(|Y(r)|_\delta + |Z_r|_\delta + |y|_\delta)\,\mathrm{d}r\right]^2 \\
&\leqslant c_3|x|_\delta^2 + \int_0^\tau \frac{c_4}{r^{2\delta}}\,\mathrm{d}r \int_0^\tau \sup_{t\in[0,\tau]} \mathbb{E}(|Y(r)|_\delta + |Z_r|_\delta + |y|_\delta)^2\,\mathrm{d}r \\
&\leqslant c_3|x|_\delta^2 + c_5\int_0^\tau \sup_{t\in[0,\tau]} \mathbb{E}|Y(r)|_\delta^2\,\mathrm{d}r + c_6|y|_\delta^2 + c_7, \quad \tau\in[0,T]
\end{aligned} \tag{2.5}$$

and hence

$$\|\Lambda(Y)\|_{\mathscr{B}}^2 = \sup_{t\in[0,T]} \mathbb{E}|\Lambda(Y)(t)|_\delta^2 \leqslant c_8(1 + \|Y\|_{\mathscr{B}}^2) \tag{2.6}$$

for some constants c_3–c_8 (possibly, depending on T). As H is linear in z, we get similarly

$$\begin{aligned}
\|\Lambda(Y_1) - \Lambda(Y_2)\|_{\mathscr{B}}^2 &= \sup_{t\in[0,T]} \mathbb{E}|\Lambda(Y_1)(t) - \Lambda(Y_2)(t)|_\delta^2 \\
&\leqslant \sup_{r\in[0,1]} |S_r|_{\mathscr{L}(H)}^2 c_\delta^2 c^2 (1-T)^{-2} \\
&\quad \times \int_0^T \frac{\mathrm{d}r}{r^{2\delta}} \int_0^T \sup_{t\in[0,T]} \mathbb{E}|(Y_1(r) - Y_2(r)|_\delta^2\,\mathrm{d}r \\
&\leqslant q_T\|Y_1 - Y_2\|_{\mathscr{B}}^2,
\end{aligned} \tag{2.7}$$

where c_δ is the norm of embedding $H_\delta \to H$ and $q_T < 1$ if T is sufficiently small. Thus, for small T, Λ is a contraction and there exists a unique solution to (1.26) in $\mathscr{B}$. For $T \in [0,1)$ large we can proceed in the usual way, dividing $[0,T]$ into small subintervals. Note that from (2.3) and (2.5) by the Gronwall lemma it follows that

$$\mathbb{E}|\hat{Z}_t|_\delta^2 \leqslant M_T(1 + |x|_\delta^2 + |y|_\delta^2), \quad x, y \in H_\delta,\ T\in[0,1), \tag{2.8}$$

where, however, the constant M_T may depend on $T \in [0,1)$ (M_T can be chosen such that $T \to M_T$ in nondecreasing).

Step II: Set $q(t,x,z) = (\mathrm{d}Q(t,x,\cdot)/\mathrm{d}\mu)(z)$. Since $Q(t,x,\cdot) = N(S_t x, Q_t)$, $\mu = N(0, Q_\infty)$, we have that

$$q(t,x,z) = \exp\{\langle Q_t^{-1/2} S_t x, Q_t^{-1/2} z\rangle - \tfrac{1}{2}|Q_t^{-1/2} S_t x|^2 + B_t(z)\}, \tag{2.9}$$

where

$$B_t(z) = \log\det(I+G)^{1/2} - \tfrac{1}{2}|G^{1/2}Q_\infty^{-1/2}z|^2,$$
$$G = (Q_t^{-1/2}Q_\infty^{1/2})^* Q_t^{-1/2}Q_\infty^{1/2} - I. \tag{2.10}$$

The fact that $Q(t,x,\cdot)$ and μ are equivalent follows from the strong Feller property, so for each $t\in\mathbb{R}_+$, $x\in H$, the density $q(t,x,\cdot)$ is well defined μ-a.e.

By the first step of the proof Eq. (1.26) has a unique solution on the interval $[0,1)$. Obviously it induces a nonhomogeneous Markov process on $[0,1)$. We will show that its transition kernels are absolutely continuous with respect to μ and their densities have the form

$$p_y(s,t,x,z) = \frac{q(t-s,x,z)q(1-t,z,y)}{q(1-s,x,y)}, \quad 0\leqslant s\leqslant t<1, \ x,y,z\in H_\delta. \tag{2.11}$$

By the Girsanov theorem we have

$$\begin{aligned}
&\mathbb{E}\varphi(\hat{Z}_{s,x}^y(t))\\
&\quad = \mathbb{E}\varphi(Z^{s,x}(t))\exp\left\{\int_s^t \langle Q^{1/2}S_{1-r}^*Q_{1-r}^{-1}y - Q^{1/2}S_{1-r}^*Q_{1-r}^{-1}S_{1-r}Z^{s,x}(r), \mathrm{d}W_r\rangle\right.\\
&\qquad \left. -\frac{1}{2}\int_s^t |Q^{1/2}S_{1-r}^*Q_{1-r}^{-1}y - Q^{1/2}S_{1-r}^*Q_{1-r}^{-1}S_{1-r}Z^{s,x}(r)|^2\mathrm{d}r\right\}
\end{aligned} \tag{2.12}$$

for $0\leqslant s\leqslant t<1$, where $\hat{Z}_{s,x}^y$ and $Z^{s,x}$ solve Eqs. (1.26) and (1.4), respectively, on the interval $[s,T)$ with initial conditions $\hat{Z}_{s,x}^y(s) = Z^{s,x}(s) = x$.

Fix $y\in\mathrm{Range}(Q_{1-t})$ where t is given by (2.11). Using well known results on strict solutions to Kolmogorov equations (see e.g. Da Prato and Zabczyk, 1992, Theorems 9.17 and 9.19) it can be seen that q satisfies

$$\frac{\partial q}{\partial r}(\sigma,x,y) = \langle D_x q(\sigma,x,y), Ax\rangle + \frac{1}{2}\operatorname{Tr} QD_x^2 q(\sigma,x,y), \quad 0<\sigma<1, \ x\in\mathrm{Dom}(A). \tag{2.13}$$

Note that

$$D_x q(1-r,x,y) = q(1-r,x,y)(S_{1-r}^*Q_{1-r}^{-1}y - S_{1-r}^*Q_{1-r}^{-1}S_{1-r}x) \tag{2.14}$$

and

$$\begin{aligned}
D_x^2 q(1-r,x,y) = {}& -q(1-r,x,y)S_{1-r}^*Q_{1-r}^{-1}S_{1-r} + q(1-r,x,y)(S_{1-r}^*Q_{1-r}^{-1}y\\
& - S_{1-r}^*Q_{1-r}^{-1}S_{1-r}x)\circ(S_{1-r}^*Q_{1-r}^{-1}y - S_{1-r}^*Q_{1-r}^{-1}S_{1-r}x),
\end{aligned} \tag{2.15}$$

while

$$D_x\log q(1-r,x,y) = S_{1-r}^*Q_{1-r}^{-1}y - S_{1-r}^*Q_{1-r}^{-1}S_{1-r}x \tag{2.16}$$

and

$$D_x^2\log q(1-r,x,y) = -S_{1-r}^*Q_{1-r}^{-1}S_{1-r}. \tag{2.17}$$

By (A.3) and the analyticity of S_t it follows that

$$\mathrm{Range}(S_rA)\subset\mathrm{Range}(Q_r), \quad r>0.$$

Since $y \in \text{Range}(Q_{1-t})$, for $y_0 = Q_{1-t}^{-1/2} y$ we have

$$|A^* S_{1-r}^* Q_{1-r}^{-1} y| \leqslant |A^* S_{1-r}^* Q_{1-r}^{-1/2}|_{\mathscr{L}(H)} \cdot |Q_{1-r}^{-1/2} Q_{1-t}^{1/2} y_0| \leqslant c_t, \tag{2.18}$$

$0 \leqslant r \leqslant t$, where c_t is a constant depending only on t because

$$|Q_{1-r}^{-1/2} Q_{1-t}^{1/2}|_{\mathscr{L}(H)} = |Q_{1-t}^{1/2} Q_{1-r}^{-1/2}|_{\mathscr{L}(H)} \leqslant 1$$

as $1-t \leqslant 1-r$ and the function $r \to |A^* S_{1-r}^* Q_{1-r}^{-1/2}|_{\mathscr{L}(H)}$ is nondecreasing for $0 \leqslant r \leqslant t$. Similarly,

$$|A^* S_{1-r}^* Q_{1-r}^{-1} S_{1-r}|_{\mathscr{L}(H)} \leqslant |A^* S_{1-r}^* Q_{1-r}^{-1/2}|_{\mathscr{L}(H)} \cdot |Q_{1-r}^{-1/2} S_{1-r}|_{\mathscr{L}(H)} \leqslant c_t, \tag{2.19}$$

$0 \leqslant r \leqslant t$.

In the sequel, let c_t denote a generic constant depending only on t. We have

$$\begin{aligned} \operatorname{Tr} S_{1-r}^* Q_{1-r}^{-1} S_{1-r} &= |Q_{1-r}^{-1/2} S_{1-r}|^2_{\mathscr{L}_2(H)} \\ &\leqslant |Q_{1-r}^{1/2}|^2_{\mathscr{L}_2(H)} \cdot |Q_{1-r}^{-1} S_{1-r}|^2_{\mathscr{L}(H)} < \infty \end{aligned} \tag{2.20}$$

for each $r > 0$ by (A.3) and

$$\operatorname{Tr} Q S_{1-r}^* Q_{1-r}^{-1} S_{1-r} \leqslant c(1-t)^{-1} |S_{1-t}|_{\mathscr{L}_1(H)} \leqslant c_t \tag{2.21}$$

for $0 \leqslant r < t$ by (A.4).

From (2.13) and estimates (2.19)–(2.21) it follows that we can apply the Itô formula to $\log q(1-r, Z_r^{s,x}, y)$ using suitable approximations of $Z^{s,x}$ by strong solutions (see e.g. Maslowski, 1995). We obtain

$$\begin{aligned} &\log q(1-t, Z^{s,x}(t), y) - \log q(1-s, x, y) \\ &\quad = \int_s^t \Big[\langle A^* S_{1-r}^* Q_{1-r}^{-1} y - A^* S_{1-r}^* Q_{1-r}^{-1} S_{1-r} Z_r^{s,x}, Z_r^{s,x} \rangle \\ &\qquad - \frac{1}{2} \operatorname{Tr} Q S_{1-r}^* Q_{1-r}^{-1} S_{1-r} - q^{-1}(1-r, Z^{s,x}(r), y) \frac{\partial}{\partial r} q(1-r, Z_r^{s,x}, y) \Big] \, \mathrm{d}r \\ &\qquad + \int_s^t \langle Q^{1/2} (S_{1-r}^* Q_{1-r}^{-1} y - S_{1-r}^* Q_{1-r}^{-1} Z^{s,x}(r)), \mathrm{d}W_r \rangle \end{aligned} \tag{2.22}$$

and by (2.13)–(2.15) we arrive at

$$\begin{aligned} &\log q(1-t, Z^{s,x}(t), y) - \log q(1-s, x, y) \\ &\quad = -\frac{1}{2} \int_s^t |Q^{1/2} (S_{1-r}^* Q_{1-r}^{-1} y - S_{1-r}^* Q_{1-r}^{-1} S_{1-r} Z^{s,x}(r))|^2 \, \mathrm{d}r \\ &\qquad + \int_s^t \langle Q^{1/2} (S_{1-r}^* Q_{1-r}^{-1} y - S_{1-r}^* Q_{1-r}^{-1} S_{1-r} Z^{s,x}(r)), \mathrm{d}W_r \rangle \end{aligned} \tag{2.23}$$

which by (2.12) yields

$$\mathbb{E} \varphi(\hat{Z}_{s,x}^y(t)) = \mathbb{E} \varphi(Z^{s,x}(t)) q(1-t, Z^{s,x}(t), y) q^{-1}(1-s, x, y) \tag{2.24}$$

for any fixed $y \in \text{Range}(Q_{1-t})$. Since $\text{Range}(Q_{1-t})$ is dense in H_δ we can extend (2.24) to all $y \in H_\delta$. Indeed, it is standard to see that the mapping $H_\delta \to H$, $y \mapsto \hat{Z}_{s,x}^y(t)$ is a.s. continuous (recall $\varphi \in C_b(H)$). On the other hand, we have

$$|\langle Q_{1-t}^{-1/2} S_{1-t} x, Q_{1-t}^{-1/2} y \rangle| = |\langle Q_{1-t}^{-1} S_{1-t} x, y \rangle| \leqslant c_t |x| \, |y| \tag{2.25}$$

and

$$|B_{1-t}(z)| \leqslant c_t + c_t|\langle Q_\infty^{-1/2}\theta_{1-t}(I-\theta_{1-t})^{-1}Q_\infty^{-1/2}z, z\rangle|, \tag{2.26}$$

$z \in \mathrm{Range}(Q_\infty^{1/2})$, where

$$\theta_t = Q_\infty^{-1/2}S_tQ_\infty(Q_\infty^{-1/2}S_t)^*$$

(cf. Fuhrman, 1996), thus we obtain

$$\begin{aligned}|B_{1-t}(z)| &\leqslant c_t + c_t|Q_\infty^{-1/2}\theta_{1-t}^{1/2}|_{\mathscr{L}(H)}|(I-\theta_{1-t})^{-1}|_{\mathscr{L}(H)}|Q_\infty^{-1/2}\theta_{1-t}^{1/2}|_{\mathscr{L}(H)}|z|^2\\ &\leqslant c_t + c_t|z|^2\end{aligned} \tag{2.27}$$

since $(I-\theta_{1-t})^{-1} \in \mathscr{L}(H)$ (cf. Fuhrman, 1996) and $Q_\infty^{-1/2}\theta_{1-t}^{1/2} \in \mathscr{L}(H)$ by (A.3). So the formula (2.24) holds for all $y \in H_\delta$ and $\varphi \in C_b(H)$, which yields (2.11).

Step III:

Set

$$H(t) = Q_tS_{1-t}^*Q_1^{-1}. \tag{2.28}$$

By (A.3) we have $H(t) \in \mathscr{L}(H)$ for $0 \leqslant t < 1$ and, obviously, $H(0) = 0$, $H(1) = \mathrm{I}$. We will check that the mapping $t \to H(t)z$ is H_δ-continuous on $[0,1]$ for each $z \in H_\delta$ and

$$|H(t)|_{\mathscr{L}(H_\delta)} \leqslant c, \quad t \in [0,1] \tag{2.29}$$

for a constant c. For $t \in (0,1)$ we have

$$Q_t = Q_1 - \int_0^{1-t} S_{r+t}QS_{r+t}^*\,\mathrm{d}r, \tag{2.30}$$

hence

$$\begin{aligned}H(t) &= Q_1S_{1-t}^*Q_1^{-1} - \left(\int_0^{1-t} S_{r+t}QS_{r+t}^*\,\mathrm{d}r\right)S_{1-t}^*Q_1^{-1}\\ &= \tilde{S}_{1-t} - S_t\left(\int_0^{1-t} S_rQS_r^*\,\mathrm{d}r\right)S_1^*Q_1^{-1}\\ &= \tilde{S}_{1-t} - S_tQ_{1-t}S_1^*Q_1^{-1}.\end{aligned} \tag{2.31}$$

From (A.5) and Pazy (1983), Theorems 6.11 and 6.12, it follows that

$$|\tilde{S}_tx - x| \leqslant k_\lambda t^\lambda|x|_\lambda, \quad \lambda \in [0,1],\ x \in H_\lambda \tag{2.32}$$

and by (A.3) it is easy to check that the mapping $[0,1] \to \mathscr{L}(H)$, $t \to S_tQ_{1-t}S_1^*Q_1^{-1}$, is λ-Hölder continuous for $\lambda < 1$, hence

$$|(H(t) - H(r))x| \leqslant k_\lambda|r-t|^\lambda|x|_\lambda, \quad x \in H_\lambda,\ r,t \in [0,1], \tag{2.33}$$

where $\lambda \in [0,1)$.

From (A.5) it follows that $t \mapsto \tilde{S}_{1-t}z$, $[0,1] \to H_\delta$ is continuous and

$$\sup_{t\in[0,1]} |\tilde{S}_{1-t}|_{\mathscr{L}(H_\delta)} < \infty.$$

Furthermore,

$$\sup_{t\in[0,1]} |S_t Q_{1-t} S_1^* Q_1^{-1}|_{\mathscr{L}(H,H_\delta)}$$

$$\leqslant c \sup_{t\in[0,1]} |S_t|_{\mathscr{L}(H)} \int_0^1 |QS_r^*|_{\mathscr{L}(H)} r^{-\delta}\,\mathrm{d}r\; |S_1^* Q_1^{-1}|_{\mathscr{L}(H)} \leqslant c \tag{2.34}$$

by (A.3) and analyticity of S_t. The proof of continuity of $t \to S_t Q_{1-t} y,\ t\in[0,1],\ y\in H$, is straightforward. Hence we obtain (2.29) and the continuity of $t \to H(t)z,\ [0,1] \to H_\delta$ for each $z\in H_\delta$. Define a stochastic process $L_t,\ t\in[0,1]$, by the equality

$$Z_t^x = S_t x + L_t + H(t)(Z_1^x - S_1 x). \tag{2.35}$$

The Ornstein–Uhlenbeck process Z_t^x evolves in $C([0,1],H_\delta)$, so by (1.6) and the continuity of $H(t)$ we get

$$\mathbb{E}|L_t|_\delta^2 \leqslant c(1+|x|_\delta^2), \quad t\in[0,1], \tag{2.36}$$

and the (Gaussian) process L_t is continuous in H_δ. Therefore, setting

$$\tilde{Z}_t := S_t x + L_t + H(t)(y - S_1 x), \quad y\in H_\delta, \tag{2.37}$$

we also have $\tilde{Z}\in C([0,1],H_\delta)$ $\mathbb{P}$-a.s. and

$$\mathbb{E}|\tilde{Z}_t|_\delta^2 \leqslant k(1+|x|_\delta^2+|y|_\delta^2), \quad t\in[0,1], \tag{2.38}$$

for some k. By (2.35) and (2.37) we have

$$\tilde{Z}_t = Z_t^x + H(t)(y - Z_1^x), \quad t\in[0,1]. \tag{2.39}$$

Note that $Z_t^x,\ t\geqslant 0$, is λ-Hölder continuous a.s. for each $\lambda<\alpha$ (cf. Seidler, 1993), which together with (2.33) and (2.39) implies the a.s. λ-Hölder continuity of $\tilde{Z}$. Furthermore, it is easy to verify that

$$\mathbb{E}|Z_t^x - Z_1^x|^2 \leqslant \tilde{C}_\delta(1+|x|_\delta^2)(1-t)^{2\delta}, \quad t\in(0,1),\ x\in H_\delta$$

holds for a constant $\tilde{C}_\delta<\infty$, hence by (2.33) and (2.39) we obtain

$$\mathbb{E}|y-\tilde{Z}_r|^2 \leqslant C_\delta(1-r)^{2\delta}(1+|y|_\delta^2+|x|_\delta^2), \quad x,y\in H_\delta,\ r\in[0,1]$$

for some constant $C_\delta<\infty$. We will prove that $\tilde{Z}_t$ is a pinned process introduced in Definition 1.1. To this end it is enough to prove that $L_t,\ t\in[0,1]$ and Z_1^x are stochastically independent (see e.g. Gikhman and Skorokhod, 1968, Lemma 3.13.2). Let $\{e_n\}$ be an orthonormal basis in H and define

$$I_t = \int_t^1 S_{1-r} Q^{1/2}\,\mathrm{d}W_r, \qquad \tilde{Q}_t = \int_t^1 S_{1-r} Q S_{1-r}^*\,\mathrm{d}r, \quad t\in[0,1].$$

Noting that $L_t = Z_t^0 - H(t)Z_1^0$ and $Z_1^0 = S_{1-t}Z_t^0 + I_t$, we obtain

$$\begin{aligned}\mathbb{E}\langle L_t,e_i\rangle\langle Z_1^0,e_j\rangle &= \mathbb{E}\langle Z_0^t - H(t)S_{1-r}Z_t^0 - H(t)I_t, e_i\rangle\langle S_{1-t}Z_t^0 + I_t, e_j\rangle\\ &= \langle -S_{1-t}Q_t S_{1-t}^* H^*(t)e_i, e_j\rangle + \langle Q_t S_{1-t}^* e_j, e_i\rangle - \langle \tilde{Q}_t H^*(t)e_i, e_j\rangle\end{aligned} \tag{2.40}$$

and using the identity $\tilde{Q}_t = Q_1 - S_{1-t}Q_tS^*_{1-t}$ we get

$$
\begin{aligned}
&\mathbb{E}\langle L_t, e_i\rangle\langle Z^0_1, e_j\rangle \\
&\quad = \langle -S_{1-t}Q_tS^*_{1-t}H^*(t)e_i + (Q_1 - S_{1-t}Q_tS^*_{1-t})H^*(t)e_i - S_{1-t}Q_te_i, e_j\rangle \\
&\quad = \langle (Q_1H^*(t) - S_{1-t}Q_t)e_i, e_j\rangle = 0, \quad i,j \in \mathbb{N}, \qquad (2.41)
\end{aligned}
$$

hence $\tilde{Z}_t$ is the pinned process from Definition 1.1. Since by (2.11) $\hat{Z}_t$ and $\tilde{Z}_t$ must have the same finite-dimension distributions, thus $\hat{Z}_t$ is a pinned process as well and (1.27), (1.28) and the λ-Hölder continuity for $\lambda < \alpha$ must be satisfied. □

For the proof of Proposition 1.2 we need the following estimate.

Lemma 2.1. *Assume* (A.1)–(A.4) *and let $x, y \in H_\delta$ and $t \in \mathbb{R}_+$ be fixed. Let $\{e_n\}$ be an orthonormal basis of H and let $g : \mathbb{R}_+ \times \Omega \to H$ be a bounded measurable function. Then, for each constant $c > 0$ there exists a constant c' such that for all $m \in \mathbb{N}$ we have*

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left\{c\sum_{n=1}^{m}\int_t^{t+1}\left|\langle g(s,w), e_n\rangle\langle Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s\right.\right.\right. \\
&\qquad \left.\left.\left. - Q^{1/2}S^*_{t+1-s}\;Q^{-1}_{t+1-s}y, e_n\rangle\right| \mathrm{d}s\right\}\right) < c', \qquad (2.42)
\end{aligned}
$$

where $(\hat{Z}_s)$, $s \in [t, t+1]$, is the pinned O–U process such that $\hat{Z}_t = x$, $\hat{Z}_{t+1} = y$, for $x, y \in H_\delta$.

Proof of Lemma 2.1. By using the Schwartz inequality and the fact that g is bounded we get that the left-hand side of (2.42) is bounded above by

$$
\mathbb{E}\left(\exp\left\{c_1\int_t^{t+1}|Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s - Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}y|\,\mathrm{d}s\right\}\right), \qquad (2.43)
$$

where c_1 is a constant. Now using Proposition 1.1(ii) with $\lambda = \delta$ and analyticity of the semigroup S_t we have by (A.4)

$$
\begin{aligned}
&|Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}(S_{t+1-s}\hat{Z}_s - y)| \\
&\quad \leqslant c(t+1-s)^{-1}(|S_{t+1-s} - I)\hat{Z}_s| + |\hat{Z}_s - y|) \\
&\quad \leqslant K_\delta[(t+1-s)^{\delta-1}|\hat{Z}_s|_\delta + (t+1-s)^{\delta-1}|\hat{Z}|_{C^\delta([t,t+1],H)}]
\end{aligned}
$$

for a suitable constant $K_\delta < \infty$. As $\hat{Z}$ is a Gaussian variable in the space $C([t,t+1], H_\delta) \cap C^\delta([t,t+1], H)$ (2.43) is finite by the Fernique inequality. □

Let $\{e_n\}$ be an orthonormal basis of H such that $\forall n \in \mathbb{N}$, $e_n \in Q^{1/2}(\mathrm{Dom}(A^*))$ (note that $Q^{1/2}(\mathrm{Dom}(A^*))$ is dense in H). For all $n \in \mathbb{N}$, $t \geqslant 0$, define $\beta_n(t) = \langle W_t, e_n\rangle$.

Proof of Proposition 1.2.

Step I: We first assume that $Q^{-1/2}f$ is bounded. Let $x \in H_\delta$ and $t \in \mathbb{R}_+$ be fixed. By the Girsanov theorem we have

$$\frac{\mathrm{d}P(t,x,t+1,\cdot)}{\mathrm{d}Q(1,x,\cdot)} = \mathbb{E}(\rho_t | Z^{t,x}_{t+1} = y),$$

$Q(1,x,\cdot)$-a.e., where

$$\rho_t := \exp\left\{ \int_t^{t+1} \langle Q^{-1/2} f(s, Z_s^{t,x}), \mathrm{d}W_s \rangle - \frac{1}{2} \int_t^{t+1} |Q^{-1/2} f(s, Z_s^{t,x})|^2 \, \mathrm{d}s \right\}$$
$$= L_1 - \lim_{m\to\infty} \rho_t^m$$

and

$$\rho_t^m := \exp\left\{ \sum_{n=1}^m \int_t^{t+1} \langle Q^{-1/2} f(s, Z_s^{t,x}), e_n \rangle \, \mathrm{d}\beta_n(s) \right.$$
$$\left. -\frac{1}{2} \int_t^{t+1} |\langle Q^{-1/2} f(s, Z_s^{t,x}), e_n \rangle|^2 \, \mathrm{d}s \right\}.$$

The sequence $\mathbb{E}(\rho^m | Z^{t,x}_{t+1} = y)$ converges in $L_1(H, \mathscr{B}(H), Q(1,x,\cdot))$ to $\mathbb{E}(\rho_t | Z^{t,x}_{t+1} = y)$ as $m \to \infty$. We denote again by $\mathbb{E}(\rho_t^m | Z^{t,x}_{t+1} = y)$ a subsequence of $\mathbb{E}(\rho_t^m | Z^{t,x}_{t+1} = y)$ converging $Q(1,x,\cdot)$-a.e.

Let Δ^k be a sequence of subdivisions of $[t, t+1]$ into k subintervals of equal length with endpoints $t = t_0 < t_1 < \cdots < t_{k-1} < t_k = t+1$, such that $|\Delta^k| \to 0$ as $k \to +\infty$. For each $n, k \in \mathbb{N}$ and $i = 0, \ldots, k-1$ define $g^n_{k,i} : C([t,t+1], H) \to \mathbb{R}$ by

$$g^n_{k,i}(\varphi) = \frac{1}{|\Delta^k|} \int_{t_{i-1}}^{t_i} \langle Q^{-1/2} f(u, \varphi(u)), e_n \rangle \, \mathrm{d}u.$$

Now, set

$$\rho_t^{m,k} := \exp\left\{ \sum_{n=1}^m \left[\sum_{i=0}^{k-1} g^n_{k,i}(Z^{t,x})(\beta_n(t_{i+1}) - \beta_n(t_i)) - \frac{1}{2} \sum_{i=0}^{k-1} |g^n_{k,i}(Z^{t,x})|^2 (t_{i+1} - t_i) \right] \right\}.$$

As $k \to \infty$, $\rho_t^{m,k}$ converges in $L^1(\Omega)$ to ρ_t^m and therefore $\mathbb{E}(\rho_t^{m,k} | Z^{t,x}_{t+1} = y)$ converges in $L_1(H, \mathscr{B}(H), Q(1,x,\cdot))$ to $\mathbb{E}(\rho_t^m | Z^{t,x}_{t+1} = y)$. We denote again by $\mathbb{E}(\rho_t^{m,k} | Z^{t,x}_{t+1} = y)$ a subsequence of $\mathbb{E}(\rho_t^{m,k} | Z^{t,x}_{t+1} = y)$ converging $Q(1,x,\cdot)$-a.e. Since the measures $Q(1,x,\cdot)$ and μ are equivalent, we get

$$\mathbb{E}(\rho_t | Z^{t,x}_{t+1} = y) = \lim_{m\to\infty} \lim_{k\to\infty} \mathbb{E}(\rho_t^{m,k} | Z^{t,x}_{t+1} = y) \quad \mu\text{-a.e.} \tag{2.44}$$

Now, for each $m, k \in \mathbb{N}$, we have

$$\mathbb{E}(\rho_t^{m,k} | Z^{t,x}_{t+1} = y) = \mathbb{E}\left(\exp\left\{ \sum_{n=1}^m \left[\sum_{i=0}^{k-1} g^n_{k,i}(Z^{t,x}) \langle Z^{t,x}_{t_{i+1}} - Z^{t,x}_{t_i}, Q^{-1/2} e_n \rangle \right. \right. \right.$$
$$- \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} g^n_{k,i}(Z^{t,x}) \langle Z_s^{t,x}, A^* Q^{-1/2} e_n \rangle \, \mathrm{d}s$$

$$
\left. - \frac{1}{2}\sum_{i=0}^{k-1} |g^n_{k,i}(Z^{t,x})|^2 \ (t_{i+1}-t_i)\Big]\Big\}\right| Z^{t,x}_{t+1} = y\Big)
$$

$$
= \mathbb{E}\Bigg(\exp\Bigg\{\sum_{n=1}^{m}\Bigg[\sum_{i=0}^{k-1} g^n_{k,i}(\hat{Z})\langle \hat{Z}_{t_{i+1}} - \hat{Z}_{t_i}, Q^{-1/2}e_n\rangle
$$

$$
-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} g^n_{k,i}(\hat{Z})\langle \hat{Z}_s, A^*Q^{-1/2}e_n\rangle \,\mathrm{d}s
$$

$$
-\frac{1}{2}\sum_{i=0}^{k-1}|g^n_{k,i}(\hat{Z})|^2(t_{i+1}-t_i)\Bigg]\Bigg\}\Bigg),
$$

where $\hat{Z}$ is the Ornstein–Uhlenbeck process conditioned to go from x at $s=t$ to y at $s=t+1$.

From (1.26) it follows that, for $i=0,\ldots,k-2$,

$$
\langle \hat{Z}_{t_{i+1}} - \hat{Z}_{t_i}, Q^{-1/2}e_n\rangle
$$

$$
= \beta_n(t_{i+1}) - \beta_n(t_i) + \int_{t_i}^{t_{i+1}} \langle \hat{Z}_s, A^*Q^{-1/2}e_n\rangle \,\mathrm{d}s
$$

$$
- \int_{t_i}^{t_{i+1}} \langle Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s - Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}y, e_n\rangle \,\mathrm{d}s.
$$

Therefore, for each $m,k\in\mathbb{N}$ we have

$$
\mathbb{E}(\rho^{m,k}_t | Z^{t,x}_{t+1} = y) = \mathbb{E}(\tilde{\Psi}_{m,k}(t,t+1,x,y)), \tag{2.45}
$$

where

$$
\tilde{\Psi}_{m,k}(t,t+1,x,y)
$$

$$
:= \exp\Bigg\{\sum_{n=1}^{m}\Bigg[\sum_{i=0}^{k-1} g^n_{k,i}(\hat{Z})(\beta_n(t_{i+1})-\beta_n(t_i)) - \frac{1}{2}\sum_{i=0}^{k-1}|g^n_{k,i}(\hat{Z})|^2(t_{i+1}-t_i)
$$

$$
-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} g^n_{k,i}(\hat{Z})\langle Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s - Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}y, e_n\rangle \,\mathrm{d}s
$$

$$
- g^n_{k,k-1}(\hat{Z})(\beta_n(t_k)-\beta_n(t_{k-1})) + g^n_{k,k-1}(\hat{Z})\langle y - \hat{Z}_{t_{k-1}}, Q^{-1/2}e_n\rangle
$$

$$
- \int_{t_{k-1}}^{t_k} g^n_{k,k-1}(\hat{Z})\langle \hat{Z}_s, A^*Q^{-1/2}e_n\rangle \,\mathrm{d}s
$$

$$
+ \int_{t_{k-1}}^{t_k} g^n_{k,k-1}(\hat{Z})\langle Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s - Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}y, e_n\rangle \,\mathrm{d}s\Bigg]\Bigg\}.
$$

Using Proposition 1.1(ii) as in the proof of Lemma 2.1 we can show that the absolute value of the sum of the last four terms in the exponent is bounded above by

$$
C_m\left[\left(\sum_{n=1}^{m}|\beta_n(t_k)-\beta_n(t_{k-1})|\right) + |y-\hat{Z}_{t_{k-1}}| + |\hat{Z}|_{C([t,t+1],H)}|t_k - t_{k-1}|\right.
$$

$$+ (|\hat{Z}|_{C([t,t+1],H_\delta)} + |\hat{Z}|_{C^\delta([t,t+1],H)}) \int_{t_{k-1}}^{t_k} (t+1-s)^{\delta-1}\,\mathrm{d}s \Bigg],$$

where C_m is a constant. Since this sum converges to zero almost surely, as $k \to \infty$, we get

$$P - \lim_{k\to\infty} \tilde{\Psi}_{m,k}(t,t+1,x,y) = \tilde{\Psi}_m(t,t+1,x,y),$$

where

$$\begin{aligned}\tilde{\Psi}_m(t,t+1,x,y) := \exp\Bigg\{\sum_{n=1}^{m}\Bigg[&\int_t^{t+1}\langle Q^{-1/2}f(s,\hat{Z}_s),e_n\rangle\,\mathrm{d}\beta_n(s)\\ &-\frac{1}{2}\int_t^{t+1}|\langle Q^{-1/2}f(s,\hat{Z}_s),e_n\rangle|^2\,\mathrm{d}s\\ &-\int_t^{t+1}\langle Q^{-1/2}f(s,\hat{Z}_s),e_n\rangle\langle Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}S_{t+1-s}\hat{Z}_s\\ &-Q^{1/2}S^*_{t+1-s}Q^{-1}_{t+1-s}y,e_n\rangle\,\mathrm{d}s\Bigg]\Bigg\}.\end{aligned}$$

It is easy to show that for each m the functions $\tilde{\Psi}_{m,k}(t,t+1,x,y)$ are uniformly integrable with respect to P, by using the Schwartz inequality, Lemma 2.1 and the fact that for each $c>0$ there exist constants c_n, c_n' and c'' such that

$$\begin{aligned}&\mathbb{E}\Bigg(\exp\Bigg\{c\,\Bigg|\,(\beta_n(t_k)-\beta_n(t_{k-1})\\ &\quad+\langle y-\hat{Z}_{t_{k-1}},Q^{-1/2}e_n\rangle-\int_{t_{k-1}}^{t_k}\langle\hat{Z}_s,A^*Q^{-1/2}e_n\rangle\mathrm{d}s\Bigg|\Bigg\}\Bigg)\\ &\quad\leqslant c_n\mathbb{E}(\exp\{c_n'|\hat{Z}|_{C([t,t+1],H)}\})\mathbb{E}(\exp\{c''|\beta_n|_{C([t,t+1],\mathbb{R})}\})\end{aligned}$$

(note that the right-hand side is finite by the Fernique inequality).

Therefore we get

$$\lim_{k\to\infty}\mathbb{E}(\tilde{\Psi}_{m,k}(t,t+1,x,y)) = \mathbb{E}(\tilde{\Psi}_m(t,t+1,x,y)). \tag{2.46}$$

As $m\to\infty$, $\tilde{\Psi}_m(t,t+1,x,y)$ converges in probability to $\Psi(t,t+1,x,y)$ given by (1.29). Since the $\tilde{\Psi}_m(t,t+1,x,y)$ are uniformly integrable (to show this we use the Schwartz inequality and the estimate in Lemma 2.1) we get

$$\lim_{m\to+\infty}\mathbb{E}(\tilde{\Psi}_m(t,t+1,x,y)) = \mathbb{E}(\Psi(t,t+1,x,y)).$$

This together with (2.44)–(2.46) implies that

$$\mathbb{E}(\rho_t|Z^{t,x}_{t+1}=y) = \mathbb{E}(\Psi(t,t+1,x,y))$$

for μ-almost all $y\in H_\delta$.

Step II: Now we remove the assumption of boundedness of $Q^{-1/2}f$ by the usual truncation method. Set

$$f_m(t,x)=\begin{cases} f(t,x), & |x|\leqslant m,\ t\in\mathbb{R}_+,\\ f\left(t,\frac{mx}{|x|}\right), & |x|\geqslant m,\ t\in\mathbb{R}_+.\end{cases}$$

It is obvious that f_m satisfies the same conditions as f for each $m\in\mathbb{N}$ and, moreover, $Q^{-1/2}f_m$ is bounded on H. Let $P_m=P_m(s,t,x,\Gamma)$, $0\leqslant s\leqslant t$, $x\in H$, $\Gamma\in\mathscr{B}(H)$, be the transition probability of the Markov process induced by the equation

$$\mathrm{d}X_t=(AX_t+f_m(t,X_t))\,\mathrm{d}t+Q^{1/2}\,\mathrm{d}W_t \tag{2.47}$$

and set

$$\begin{aligned}\Psi_m(s,t,x,y)&\\ :=\exp\Bigg\{&\int_s^t\langle Q^{-1/2}f_m(r,\hat Z_r),\mathrm{d}W_r\rangle-\frac12\int_s^t|Q^{-1/2}f_m(r,\hat Z_r)|^2\,\mathrm{d}r\\ &-\int_s^t\langle Q^{-1/2}f_m(r,\hat Z_r),Q^{1/2}S^*_{1+s-r}Q^{-1}_{1+s-r}S_{1+s-r}\hat Z_r-Q^{1/2}S^*_{1+s-r}y\rangle\,\mathrm{d}r\Bigg\}.\end{aligned}$$

By the preceding part of the proof we have

$$\frac{\mathrm{d}P_m(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y)=\mathbb{E}\Psi_m(t,t+1,x,y) \tag{2.48}$$

for $Q(1,x,\cdot)$-almost all $y\in H_\delta$. Since the paths of the Ornstein–Uhlenbeck process $Z^{t,x}$ belong to $C([t,t+1];H)$ and f_m coincide with f on arbitrarily large balls in H, for m large we easily get

$$\|P_m(t,t+1,x,\cdot)-P(t,t+1,x,\cdot)\|_{\mathrm{var}}\to 0,\quad m\to\infty \tag{2.49}$$

for each $t\in\mathbb{R}_+$, $x\in H$, and hence

$$\frac{\mathrm{d}P_m(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}\to\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}\quad\text{in } L_1(Q(1,x,\cdot)). \tag{2.50}$$

Therefore, for a subsequence (denoted again by P_m) we have

$$\frac{\mathrm{d}P_m(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y)\to\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y),\quad m\to\infty \tag{2.51}$$

for $Q(1,x,\cdot)$-almost all $y\in H_\delta$. On the other hand, it is straightforward to verify that

$$\Psi_m(t,t+1,x,y)\to\Psi(t,t+1,x,y),\quad\mathbb{P}\text{-a.s.} \tag{2.52}$$

for each $t\in\mathbb{R}_+$, $x,y\in H_\delta$ (possibly, for a subsequence). Now, (2.48), (2.51) and (2.52) yield

$$\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y)\geqslant\mathbb{E}\Psi(t,t+1,x,y) \tag{2.53}$$

for $Q(1,x,\cdot)$-almost all $y\in H_\delta$ by virtue of the Fatou lemma.

Proof of Proposition 1.3. From Proposition 1.2 and the Jensen inequality it follows that

$$\begin{aligned}
&\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y)\\
&\geqslant \exp\left\{\mathbb{E}\left[\int_t^{t+1}\langle Q^{-1/2}f(r,\hat{Z}_r),\mathrm{d}W_r\rangle - \frac{1}{2}\int_t^{t+1}|Q^{-1/2}f(r,\hat{Z}_r)|^2\,\mathrm{d}r\right.\right.\\
&\quad - \int_t^{t+1}\langle Q^{-1/2}f(r,\hat{Z}_r), Q^{1/2}S^*_{t+1-r}Q^{-1}_{t+1-r}S_{t+1-r}\hat{Z}_r\\
&\quad \left.\left. - Q^{1/2}S^*_{t+1-r}Q^{-1}_{t+1-r}y\rangle\ \mathrm{d}r\right]\right\}\\
&\geqslant \exp\left\{-\mathbb{E}\int_t^{t+1}\left[\frac{1}{2}|Q^{-1/2}f(r,\hat{Z}_r)|^2 + |\langle Q^{-1/2}f(r,\hat{Z}_r),\right.\right.\\
&\quad \left.\left. Q^{1/2}S^*_{t+1-r}Q^{-1}_{t+1-r}S_{t+1-r}\hat{Z}_r - Q^{1/2}S^*_{t+1-r}Q^{-1}_{t+1-r}y\rangle|\right]\mathrm{d}r\right\}\ \text{a.e.}
\end{aligned}\tag{2.54}$$

By (A.1), (A.4), Proposition 1.1 and the analyticity of the semigroup S_t we have

$$\begin{aligned}
&\mathbb{E}\int_t^{t+1}\left[\frac{1}{2}|Q^{-1/2}f(r,\hat{Z}_r)|^2\right.\\
&\quad \left. + |\langle Q^{-1/2}f(r,\hat{Z}_r), Q^{1/2}S^*_{t+1-r}Q^{-1}_{t+1-r}(S_{t+1-r}\hat{Z}_r - y)\rangle|\right]\mathrm{d}r\\
&\leqslant \mathbb{E}\int_t^{t+1}\frac{K^2}{2}(1+|\hat{Z}_r|_\delta)^2\,\mathrm{d}r\\
&\quad + \mathbb{E}\int_t^{t+1}K(1+|\hat{Z}_r|_\delta)c(t+1-r)^{-1}(|(S_{t+1-r}-I)\hat{Z}_r| + |\hat{Z}_r - y|)\,\mathrm{d}r\\
&\leqslant K^2 + K^2k^2(1+|x|^2_\delta + |y|^2_\delta)\\
&\quad + \mathbb{E}\int_t^{t+1}K(1+|\hat{Z}_r|_\delta)C_\delta(t+1-r)^{\delta-1}(|\hat{Z}_r|_\delta + |\hat{Z}_r - y|(t+1-r)^{-\delta})\,\mathrm{d}r\\
&\leqslant K^2 + K^2k^2(1+|x|^2_\delta + |y|^2_\delta) + C'_\delta\int_t^{t+1}(t+1-r)^{\delta-1}(1+|x|_\delta + |y|_\delta)\,\mathrm{d}r\\
&\quad + C'_\delta\int_t^{t+1}(t+1-r)^{\delta-1}[(1+|x|^2_\delta + |y|^2_\delta) + \mathbb{E}|\hat{Z}_r|^2_\delta\\
&\quad + (t+1-r)^{-\delta}(\mathbb{E}|\hat{Z}_r|^2_\delta)^{1/2}\times(\mathbb{E}|\hat{Z}_r - y|^2)^{1/2}]\,\mathrm{d}r\\
&\leqslant C''_\delta(1+|x|^2_\delta + |y|^2_\delta),\quad x,y\in H_\delta
\end{aligned}$$

for suitable constants C'_δ, C''_δ, which implies

$$\frac{\mathrm{d}P(t,t+1,x,\cdot)}{\mathrm{d}Q(1,x,\cdot)}(y) \geqslant k_1\exp\{-k_1(|x|^2_\delta + |y|^2_\delta)\}\tag{2.55}$$

for $x\in H_\delta$ and $Q(1,x,\cdot)$-almost all $y\in H_\delta$ where k_1 is a suitable constant.

By (2.9) we have

$$\frac{\mathrm{d}Q(1,x,\cdot)}{\mathrm{d}\mu}(y) = \exp\left\{\langle Q^{-1/2}S_1x, Q_1^{-1/2}y\rangle - \frac{1}{2}|Q_1^{-1/2}S_1x|^2 + B_1(y)\right\}$$
$$\geqslant k_2\exp\{-k_2(|x|_\delta^2 + |y|_\delta^2)\} \tag{2.56}$$

for a $k_2 > 0$ and all $x, y \in H_\delta$, since $Q_1^{-1/2}S_1 \in \mathscr{L}(H)$ by the assumption (A.3), and $|B_1(y)| \leqslant c_1(1+|y|^2)$ by (2.27). From (2.55) and (2.56) we obtain (1.31). □

In order to prove Theorem 1.4 we need to verify a kind of ultimate boundedness of the second moment in $|\cdot|_\delta$ in the form stated in the following lemma.

Lemma 2.2. *Let assumptions* (A.1), (A.2) *and* (A.6) *be satisfied. Then there exists a constant $M < \infty$ such that for each $x \in H$, $s \geqslant 0$, we have*

$$\mathbb{E}_{s,x}|X_t|_\delta^2 \leqslant M \tag{2.57}$$

for all $t \geqslant t_0$, where $t_0 \geqslant s$ may depend on s and x.

Proof of Lemma 2.2. *Step I*: At first we prove the assertion with the H-norm replacing $|\cdot|_\delta$ that is, we will show

$$\mathbb{E}_{s,x}|X_t|^2 \leqslant M_1 \tag{2.58}$$

for some $M_1 < \infty$ and all $t \geqslant t_0(s,x)$. For fixed $s \geqslant 0$, $x \in H$, there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and a standard cylindrical Wiener process $\tilde{W}_t$ on H defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ such that

$$X_t = S_{t-s}x + \int_s^t S_{t-r}f(r, X_r)\,\mathrm{d}r + \tilde{Z}_t, \quad t \geqslant s, \tag{2.59}$$

where

$$\tilde{Z}_t = \int_s^t S_{t-r}Q^{1/2}\,\mathrm{d}\tilde{W}_r.$$

Setting $Y(t) = X_t - \tilde{Z}_t$ we have that

$$Y(t) = S_{t-s}x + \int_s^t S_{t-r}f(r, Y(r) + \tilde{Z}_r)\,\mathrm{d}r. \tag{2.60}$$

The operator $A_\lambda := \lambda^2(\lambda I - A^*)^{-1}A(\lambda I - A)^{-1}$ is bounded on H for every $\lambda > 0$ and from the properties of the Yosida approximations it follows that $A_\lambda y \to Ay$, as $\lambda \to \infty$, for $y \in \mathrm{Dom}(A)$. Therefore, denoting by S_t^λ the semigroup on H generated by A_λ, we obtain $\sup_{t\in[0,T]}|S_t^\lambda y - S_t y| \to 0$ as $\lambda \to \infty$ (cf. Pazy, 1983, Theorem 3.4.5). It is also obvious that S^λ is a semigroup of contractions. Hence for an approximating sequence Y_λ defined by

$$Y_\lambda(t) = S_{t-s}^\lambda x + \int_s^t S_{t-r}^\lambda f(r, Y(r) + \tilde{Z}_r)\,\mathrm{d}r, \quad t \geqslant s,\ \lambda > 0, \tag{2.61}$$

we have

$$|Y_\lambda(r) - Y(r)| \to 0, \quad r \geqslant s,\ \lambda \to \infty, \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{2.62}$$

and

$$\sup_{r\in[s,T],\lambda>0} |Y_\lambda(r)| < \infty, \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{2.63}$$

for each $T > s$. Since the operators A_λ are bounded, Eq. (2.61) has a strong solution

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_\lambda(t) = A_\lambda Y_\lambda(t) + f(t, Y(t) + \tilde{Z}_t), \quad t \geqslant s,$$

$$Y_\lambda(s) = x \tag{2.64}$$

and we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Y_\lambda(t)|^2 &= \langle A_\lambda Y_\lambda(t), Y_\lambda(t)\rangle + \langle f(t, Y(t) + \tilde{Z}_t), Y_\lambda(t)\rangle \\ &\leqslant -\omega|\lambda(\lambda I - A)^{-1} Y_\lambda(t)|^2 + \langle f(t, Y(t) + \tilde{Z}_t), Y_\lambda(t)\rangle \\ &\leqslant -\omega|Y_\lambda(t)|^2 + k(t)|Y(t)|^2 + a(t, |\tilde{Z}_t|) + \omega(|Y_\lambda(t)| - |Y(t)|) \\ &\quad + |\langle f(t, Y(t) + \tilde{Z}_t), Y_\lambda(t) - Y(t)\rangle| \\ &\leqslant -\omega|Y_\lambda(t)|^2 + k(t)|Y_\lambda(t)|^2 + a(t, \tilde{Z}_t) + \delta_\lambda(t), \quad \tilde{\mathbb{P}}\text{-a.s.},\end{aligned} \tag{2.65}$$

where

$$\delta_\lambda(t) = (\omega + |k(t)|)(|Y_\lambda(t)|^2 - |Y(t)|^2) + |f(t, Y(t) + \tilde{Z}_t)|\,|Y_\lambda(t) - Y(t)|. \tag{2.66}$$

It follows that

$$\begin{aligned}|Y_\lambda(t)|^2 &\leqslant |x|^2 \exp\left\{-\int_s^t (\tilde{\omega}(\lambda)\,\mathrm{d}\lambda\right\} \\ &\quad + \int_s^t \exp\left\{-\int_r^t \tilde{\omega}(\lambda)\,\mathrm{d}\lambda\right\}(a(r, |\tilde{Z}_r|) + \delta_\lambda(r))\,\mathrm{d}r, \quad t \geqslant s\end{aligned} \tag{2.67}$$

and since $\delta_\lambda(t) \to 0$ as $\lambda \to \infty$ and $\sup_{t\in[s,T],\lambda>0} \delta_\lambda(t) < \infty$ $\tilde{\mathbb{P}}$-a.s. by (2.62), (2.63) and (A.1), we arrive at

$$\begin{aligned}\mathbb{E}_{s,x}|Y(t)|^2 &\leqslant |x|^2 \exp\left\{-\int_s^t \tilde{\omega}(r)\,\mathrm{d}r\right\} \\ &\quad + \mathbb{E}_{s,x}\int_s^t \exp\left\{-\int_r^t \tilde{\omega}(\lambda)\,\mathrm{d}\lambda\right\} a(r, |\tilde{Z}_r|)\,\mathrm{d}r.\end{aligned} \tag{2.68}$$

The process $\tilde{Z}$ has the same probability law as ϕ and clearly

$$\sup_{t\geqslant s} \mathbb{E}_{s,x}|\tilde{Z}_t|^2 = \sup_{t\geqslant s} \mathrm{Tr}\int_s^t S_r Q S_r^*\,\mathrm{d}r < \infty \tag{2.69}$$

as follows from (A.2) and the stability of S_r. Taking into account assumption (A.6) we conclude the proof of (2.58).

Step II: For $t \geqslant s$ we have

$$X_{t+1} = S_1 X_t + \int_t^{t+1} S_{t+1-r} f(r, X_r)\,\mathrm{d}r + \int_t^{t+1} S_{t+1-r} Q^{1/2}\,\mathrm{d}\tilde{W}_r.$$

By a standard regularity result (see e.g. Da Prato and Zabczyk, 1992) and (1.5) we have

$$\sup_{t\geqslant 0}\mathbb{E}_{s,x}\left|\int_t^{t+1} S_{t+1-r}Q^{1/2}\,\mathrm{d}\tilde{W}_r\right|_\delta^2 = \int_0^1 |S_rQ^{1/2}|^2_{\mathscr{L}_2(H,H_\delta)}\,\mathrm{d}r =: L < \infty \tag{2.70}$$

and analyticity and contractivity of the semigroup together with the assumption (A.1) yield

$$\begin{aligned}\mathbb{E}_{s,x}|X_{t+1}|_\delta^2 &\leqslant C\left(\mathbb{E}_{s,x}|X_t|^2 + \int_t^{t+1}\frac{\mathbb{E}_{s,x}(1+|X_r|^2)}{(t+1-r)^{2\delta}}\,\mathrm{d}r + L\right)\\ &\leqslant C\left(M_1 + (M_1+1)\int_0^1\frac{\mathrm{d}r}{r^{2\delta}} + L\right), \quad t\geqslant t_0,\end{aligned} \tag{2.71}$$

for a universal constant C which concludes the proof. □

Remark 2.2. For $f: H\to H$ continuous an assertion similar to the one contained in step I above can be found in Maslowski and Simão (1997, Proposition 2.4). However, in Maslowski and Simão (1997) the continuity of f is needed in the proof.

Proof of Theorem 1.4. Let $B_\delta(r)$, $0<\delta<\frac{1}{2}$, $r>0$, denote the ball in H_δ, $B_\delta(r) := \{y\in H_\delta, |y|_\delta \leqslant r\}$. By Lemma 2.1 we have

$$\mathbb{P}_{s,x}[X_t\notin B_\delta(r)] \leqslant \frac{\mathbb{E}_{s,x}|X_t|_\delta^2}{r^2} \leqslant \frac{M}{r^2}, \quad t\geqslant t_0(s,x),$$

so for a fixed $r_0 > \sqrt{M}$ we get

$$P(s,t,x,B_\delta(r_0)) \geqslant \lambda := 1 - \frac{M}{r_0^2}, \quad t\geqslant t_0(s,x). \tag{2.72}$$

Furthermore, by Proposition 1.3, we have

$$\begin{aligned}\inf_{y\in B_\delta(r_0)} P(t,t+1,y,\Gamma) &= \inf_{y\in B_\delta(r_0)}\int_\Gamma h_t(y,z)\mu(\mathrm{d}z)\\ &\geqslant \inf_{y\in B_\delta(r_0)}\int_\Gamma k_2\exp\{-k_2(|y|_\delta^2 + |z|_\delta^2)\mu(\mathrm{d}z) \geqslant k_2\mathrm{e}^{-k_2r_0^2}\}\bar{\mu}(\Gamma), \quad \Gamma\in B,\end{aligned} \tag{2.73}$$

where

$$\bar{\mu}(\Gamma) := \int_\Gamma \mathrm{e}^{-k_2|z|_\delta^2}\mu(\mathrm{d}z).$$

Thus for each $v\in\mathscr{P}$ and $\Gamma\in\mathscr{B}$ we get

$$\begin{aligned}[P^*_{s,t+1}v](\Gamma) &= \int_H\int_H P(t,t+1,y,\Gamma)P(s,t,x,\mathrm{d}y)v(\mathrm{d}x)\\ &\geqslant \int_H\int_{B_\delta(r_0)} P(t,t+1,y,\Gamma)P(s,t,x,\mathrm{d}y)v(\mathrm{d}x)\\ &\geqslant \lambda k_2\mathrm{e}^{-k_2r_0^2}\bar{\mu}(\Gamma), \quad t\geqslant t_0(s,x).\end{aligned} \tag{2.74}$$

Therefore, the nonnegative (and nonzero) measure $\hat{\mu} := \lambda k_2\mathrm{e}^{-k_2r_0^2}\bar{\mu}$ satisfies

$$\|(P^*_{s,t}v - \hat{\mu})^-\|_{\mathrm{var}} \to 0, \quad t\to\infty \tag{2.75}$$

for each $s \in \mathbb{R}_+$ and $v \in \mathscr{P}$, where v^- denotes the negative variation of a measure v, so $\hat{\mu}$ is a lower bound measure for the system $P^*_{s,t}$, and (1.33) and (1.34) follow (cf. Lasota and Mackey, 1994; Maslowski and Simão, 1997). □

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